Final Exam — Analysis (WBMA012-05)

Wednesday 2 February 2022, 16.00h–18.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. Provide clear arguments for all your answers: only answering "yes", "no", or "42" is not sufficient. You may use all theorems and statements in the book, but you should clearly indicate which of them you are using.
3. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (10 + 6 + 4 + 4 = 24 points)

Let (x_n) be a bounded sequence and consider the sequences (a_n) and (b_n) defined by

$$a_n = \inf\{x_k \,:\, k \geq n\} \quad \text{and} \quad b_n = \sup\{x_k \,:\, k \geq n\}.$$

(a) Use the Monotone Convergence Theorem to prove that the sequences (a_n) and (b_n) are convergent.

Hint: recall that $A \subseteq B$ implies that $\sup A \leq \sup B$ and $\inf B \leq \inf A$.

(b) Let $a = \lim a_n$ and $b = \lim b_n$. Show that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Rightarrow \quad a - \epsilon < a_n \leq x_n \leq b_n < b + \epsilon.$$

(c) Assume that $a = b$. Show that the sequence (x_n) is convergent.

(d) Is the sequence (x_n) still convergent when $a < b$? If so, give a proof; otherwise, give a counterexample.

Problem 2 (7 + 7 + 7 = 21 points)

Consider the following set:

$$A = \left\{ \frac{1}{p} + \frac{1}{q} \,:\, p, q \in \mathbb{N} \right\}.$$

Show that A is *not* compact in three different ways:

(a) A does not satisfy the definition of a compact set;

(b) A is not closed;

(c) A has an open cover without a finite subcover.

Problem 3 (5 + 6 + 5 + 6 = 22 points)

Let $f : [-1, 2] \to \mathbb{R}$ be a differentiable function. Assume that

$$f(-1) = 4, \quad f(0) = -1, \quad \text{and} \quad f'(x) \geq 3 \quad \text{for all} \quad x \in [0, 2].$$

Prove the following statements:

(a) $f(2) \geq 5$.

(b) f has at least two zeros on the interval $(-1, 2)$.

(c) $f'(s) = 0$ for some $s \in (-1, 2)$.

(d) $f'(t) = \sqrt{7}$ for some $t \in (-1, 2)$.

Problem 4 (5 + 9 + 9 = 23 points)

Let $g : [0, 1] \to \mathbb{R}$ be a continuous function and consider the sequence (f_n) given by

$$f_n : [0, 1] \to \mathbb{R}, \quad f_n(x) = g(x^n).$$

Prove the following statements:

(a) The sequence (f_n) converges pointwise to $f : [0, 1] \to \mathbb{R}$ where

$$f(x) = \begin{cases} g(0) & \text{if } 0 \leq x < 1, \\ g(1) & \text{if } x = 1. \end{cases}$$

(b) If the convergence $f_n \to f$ is uniform on $[0, 1]$, then $g(0) = g(1)$.

(c) The converse of part (b) is not true.
Hint: consider the function $g(x) = \sin(\pi x)$.

End of test (90 points)

Solution of Problem 1 (10 + 6 + 4 + 4 = 24 points)

(a) Consider the set $S_n = \{x_k : k \geq n\}$. Clearly, we have that $S_{n+1} \subseteq S_n$, which implies that

$$\inf S_n \leq \inf S_{n+1} \quad \text{and} \quad \sup S_{n+1} \leq \sup S_n,$$

or, equivalently,

$$a_n \leq a_{n+1} \quad \text{and} \quad b_{n+1} \leq b_n.$$

This shows that both (a_n) and (b_n) are monotone sequences.
(2 points)

It is given that the sequence is bounded, which means that there exists a constant $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. In other words, we have that $-M \leq x_n \leq M$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be arbitrary. We have that $a_n \leq x_n \leq M$. We also have that $-M \leq x_k$ for all $k \geq n$ (indeed, we have the latter inequality for *all* $n \in \mathbb{N}$), but since a_n is the *greatest* lower bound of the set $S_n = \{x_k : k \geq n\}$ it follows that $-M \leq a_n$. This shows that $|a_n| \leq M$, which means that the sequence (a_n) is bounded.
(3 points)

We have that $-M \leq x_n \leq b_n$. We also have that $x_k \leq M$ for all $k \geq n$ (indeed, we have the latter inequality for *all* $n \in \mathbb{N}$), but since b_n is the *least* upper bound of the set $S_n = \{x_k : k \geq n\}$ it follows that $b_n \leq M$. This shows that $|b_n| \leq M$, which means that the sequence (b_n) is bounded.
(3 points)

Since the sequences (a_n) and (b_n) are bounded and monotone it follows by the Monotone Convergence Theorem that they are convergent.
(2 points)

(b) Let $a = \lim a_n$ and $b = \lim b_n$. Let $\epsilon > 0$ be arbitrary. There exists $N_1 \in \mathbb{N}$ such that

$$n \geq N_1 \quad \Rightarrow \quad |a_n - a| < \epsilon \quad \Rightarrow \quad -\epsilon < a_n - a < \epsilon \quad \Rightarrow \quad a - \epsilon < a_n.$$

(2 points)

Likewise, there exists $N_2 \in \mathbb{N}$ such that

$$n \geq N_2 \quad \Rightarrow \quad |b_n - b| < \epsilon \quad \Rightarrow \quad -\epsilon < b_n - b < \epsilon \quad \Rightarrow \quad b_n < b + \epsilon.$$

(2 points)

By definition of the sequences (a_n) and (b_n) we have that $a_n \leq x_n \leq b_n$ for all $n \in \mathbb{N}$. For $N = \max\{N_1, N_2\}$ combining all inequalities gives

$$n \geq N \quad \Rightarrow \quad a - \epsilon < a_n \leq x_n \leq b_n < b + \epsilon.$$

(2 points)

(c) Assume that $a = b$. Let $\epsilon > 0$ be arbitrary. Part (b) implies that there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Rightarrow \quad a - \epsilon < x_n < a + \epsilon \quad \Rightarrow \quad |x_n - a| < \epsilon.$$

By definition, this means that the sequence (x_n) is convergent and $\lim x_n = a$. (Of course, we could also say that $\lim x_n = b$ since it was assumed that $a = b$.)
(4 points)

(d) If $a < b$, then the sequence (x_n) is not necessarily convergent. As a counterexample, take $x_n = (-1)^n$. Then $a_n = -1$ and $b_n = 1$ for all $n \in \mathbb{N}$, which immediately implies that $a = -1$ and $b = 1$ so that $a < b$. The sequence (x_n) is not convergent since we can make two convergent subsequences that have different limits (just consider terms with even or odd indices n).
(4 points)

Solution of Problem 2 (7 + 7 + 7 = 21 points)

(a) By taking $p = q = n$ we obtain the sequence (a_n) in the set A which is given by $a_n = 2/n$. Since (a_n) converges to zero, every subsequence also converges to zero.
(4 points)

However, $0 \notin A$. Therefore, we have showed the existence of a sequence which does not have a convergent subsequence such that the limit of that subsequence is contained in the set A. It follows from Definition 3.3.1 that A is not compact.
(3 points)

(b) Note that 0 is a limit point of A. Indeed, with $a_n = 2/n$ we have a sequence in A such that $a_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim a_n = 0$. By Theorem 3.2.5 it follows that 0 is a limit point of A.
(5 points for showing that 0 is a limit point of A)

However, $0 \notin A$, so by Definition 3.2.7 it follows that A is not closed. Finally, Theorem 3.3.8 implies that A is not compact.
(2 points for the conclusion)

(c) First note that every element of A lies in the interval $(0, 2]$. The sets

$$O_n = \left(\frac{2}{n}, 3\right) \quad \text{with} \quad n \in \mathbb{N}$$

are obviously open. In addition

$$\bigcup_{n\in\mathbb{N}} O_n = (0, 3) \supset A,$$

which implies that the sets O_n form an open cover for A.
(3 points)

On the other hand, for indices $n_1, n_2, \ldots, n_k \in \mathbb{N}$ we have

$$O_{n_1} \cup O_{n_2} \cup \cdots \cup O_{n_k} = \left(\frac{2}{m}, 3\right),$$

where $m = \max\{n_1, \ldots, n_k\}$. The right hand side does *not* contain the set A. Indeed, we already know that 0 is a limit point of A so the set A has elements arbitrarily close to 0. This means that the open cover given by the sets O_n does not have a finite subcover.
(4 points)

Solution of Problem 3 (5 + 6 + 5 + 6 = 22 points)

(a) Since the function f is differentiable on $(0, 2)$ and continuous on $[0, 2]$, the Mean Value Theorem implies that there exists a point $c \in (0, 2)$ such that

$$\frac{f(2) - f(0)}{2 - 0} = f'(c),$$

(3 points)

Rearranging terms and using the assumption that $f'(c) \geq 3$ gives

$$f(2) = f(0) + 2f'(c) = -1 + 2f'(c) \geq -1 + 2 \cdot 3 = 5.$$

(2 points)

(b) Since $f(-1) > 0$ and $f(0) < 0$, the Intermediate Value Theorem implies that there exists a point $z_1 \in (-1, 0)$ such that $f(z_1) = 0$.
(3 points)

Since $f(0) < 0$ and $f(2) > 0$, the Intermediate Value Theorem implies that there exists a point $z_2 \in (0, 2)$ such that $f(z_2) = 0$.
(3 points)

This shows that f indeed has at least two zeros on the interval $(-1, 2)$.

(c) We can show this statement in two different ways.

Method 1. Since part (b) implies the existence of two points $z_1, z_2 \in (-1, 2)$ with $z_1 < z_2$ and $f(z_1) = f(z_2)$, it follows by Rolle's Theorem that there exists a point $s \in (z_1, z_2)$ such that $f'(s) = 0$. (The Mean Value Theorem can also be used here.)
(5 points)

Method 2. Since f is continuous on the compact set $[-1, 2]$ it follows that f attains a minimum value at a point $s \in [-1, 2]$. Since $f(-1) > 0$ and $f(2) > 0$ but $f(0) < 0$ it follows that the minimum cannot be attained at the boundary points of the interval $[-1, 2]$. Hence, $s \in (-1, 2)$. By the Interior Extremum Theorem it then follows that $f'(s) = 0$.
(5 points)

(d) By the previous argument, we have a point $s \in (-1, 2)$ such that $f'(s) = 0$. By assumption we have that $f'(2) = 3$. Since $0 < \sqrt{7} < 3$, it follows by Darboux's Theorem that there exists a point $t \in (s, 2) \subset (-1, 2)$ such that $f'(t) = \sqrt{7}$.
(6 points)

Note: we *cannot* apply the Intermediate Value Theorem to f' since it is only given that f is differentiable, but not that f' is a continuous function!
(only 3 points when IVT has been applied)

Solution of Problem 4 (5 + 9 + 9 = 23 points)

(a) We have that $f_n(1) = g(1)$ for all $n \in \mathbb{N}$ so that trivially $\lim f_n(1) = g(1)$.
(1 point)

For $0 \le x < 1$ we have that $\lim x^n = 0$. By continuity of g it follows that

$$\lim f_n(x) = \lim g(x^n) = g(0).$$

(4 points)

(b) The function $h_n(x) = x^n$ is continuous as a product of continuous functions. The function g is continuous on $[0, 1]$ by assumption. Since compositions of continuous functions are again continuous, it follows that each function $f_n = g \circ h_n$ is continuous on $[0, 1]$.
(3 points)

If the convergence $f_n \to f$ is uniform on $[0, 1]$, then the limit function f is also continuous on $[0, 1]$.
(2 points)

To show that $g(0) = g(1)$ we can use different arguments.

Method 1. The sequential characterization of continuity implies that that for any convergent sequence (c_n) with $\lim c_n = 1$ we have that $\lim f(c_n) = f(1) = g(1)$. In particular, this must hold for the sequence $c_n = 1 - 1/n$ for which we have that $f(c_n) = g(0)$ for all $n \in \mathbb{N}$ and thus $\lim f(c_n) = g(0)$. By uniqueness of limits, it follows that $g(0) = g(1)$.
(4 points)

Method 2. In particular, the function f is continuous at $x = 1$. By definition, this means that for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x - 1| < \delta \quad \Rightarrow \quad |f(x) - f(1)| < \epsilon.$$

Now let $0 < \delta < 1$ and take any $x \in (1-\delta, 1)$. For example, we could take $x = 1-\delta/2$. The implication above then gives

$$|g(0) - g(1)| < \epsilon,$$

where we have used the formula for f obtained in part (a). Since this inequality holds for all $\epsilon > 0$, it follows that $g(0) = g(1)$.
(4 points)

Method 3. Assume that $g(0) \neq g(1)$. Then $\epsilon = \frac{1}{2}|g(0) - g(1)| > 0$. Since f is continuous at $x = 1$, there exists $\delta > 0$ such that

$$|x - 1| < \delta \quad \Rightarrow \quad |f(x) - f(1)| < \epsilon.$$

Now let $0 < \delta < 1$ and take any $x \in (1-\delta, 1)$. For example, we could take $x = 1-\delta/2$. The implication above then gives

$$|g(0) - g(1)| < \epsilon = \tfrac{1}{2}|g(0) - g(1)|,$$

where we have used the formula for f obtained in part (a). This is obviously a contradiction. Hence, we must conclude that $g(0) = g(1)$.
(4 points)

(c) For $g(x) = \sin(\pi x)$ we clearly have that $g(0) = g(1) = 0$ so that in particular $f = 0$. However, the convergence $f_n \to f$ is not uniform on $[0, 1]$ as is shown next.

By elementary calculus methods, it follows that the functions f_n attain their maximum value at $x_n = \sqrt[n]{1/2}$. Therefore, we have for all $n \in \mathbb{N}$ that

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} f_n(x) = f_n(x_n) = \sin(\pi/2) = 1$$

(5 points)

In particular, we do *not* have that

$$\lim \left(\sup_{x\in[0,1]} |f_n(x) - f(x)| \right) = 0.$$

This shows that the convergence $f_n \to f$ is not uniform on $[0, 1]$.
(4 points)